THE BELL SYSTEM
TECHNICAL JOURNAL

DEVOTED TO THE SCIENTIFIC AND ENGINEERING
ASPECTS OF ELECTRICAL COMMUNICATION

Volume 52 | October 1973 | Number 8

Volterra Systems With More Than One Input Port—Distortion in a Frequency Converter

By S. O. RICE

(Manuscript received April 19, 1973)

Consider a nonlinear system, with memory, which has two input ports and one output port. It is assumed that the system can be represented by a double Volterra series. Two results for such a system are stated in Part I. The first is a general expression for the sinusoidal components of the output $y(t)$ when the two inputs $x_u(t)$ and $x_v(t)$ are sums of sinusoidal terms. The second result is an expression for the power spectrum of $y(t)$ when $x_u(t)$ is a stationary Gaussian process and $x_v(t) = P \cos pt$. Part II is concerned with using results from the theory of Volterra series for multi-input systems to calculate the third-order distortion in an idealized frequency converter.

I. INTRODUCTION

This paper deals with nonlinear, time-invariant systems with memory which (*i*) have more than one input port, and (*ii*) are driven by inputs which are essentially sums of sine waves.

The paper consists of two parts. Part I is concerned with a system which has two inputs, $x_u(t)$ and $x_v(t)$, and one output $y(t)$. Two results for single-input systems are generalized: (*i*) an expression is given for an arbitrary frequency component of $y(t)$ when $x_u(t)$ and $x_v(t)$ are

finite sums of sine waves, and (*ii*) an expression is given for the power spectrum of $y(t)$ when $x_u(t)$ is a stationary, zero-mean, Gaussian noise, and $x_v(t)$ is a single sine wave.

Although most of the discussion in Part I deals with systems having two input ports, many of the results can be formally generalized to systems with more than two inputs.

Part II is devoted to an example which shows how results given in Ref. 1 for a one-input Volterra system can be used to examine systems consisting of a single two-terminal nonlinear element imbedded in a linear network containing sources. The transformation from a multi-input to a single-input system is based upon Thévenin's theorem (see, for example, Anderson and Leon[2]). The example treated here is a frequency converter using a nonlinear capacitor. Particular attention is paid to computing the limiting form of the expression for the third-order distortion when the signal and pump amplitudes become small.

The procedure we use in Part II is essentially a systemization of a procedure used by Gardiner and Ghobrial[3] to study the distortion performance of a varactor frequency converter. As they point out, their treatment differs from the linear time-varying analysis usually employed to study frequency converters. It is appropriate to mention here that a promising new general method of computing distortion in frequency converters has been developed by R. B. Swerdlow.[4] His method is based upon the use of Volterra series with time-varying kernels.

Part I. Two Input Ports

When analysis of the type used to study Volterra systems is applied to nonlinear circuits having two input ports and one output port, some of the simpler results for one-input circuits can be generalized in a straightforward way. Here we state two such generalizations. In the first, the two inputs are sums of sine waves. In the second, one input is stationary, zero-mean Gaussian noise, and the other input is a single sine wave.

The derivations of the generalizations are not given here because they consist of rather straightforward, although lengthy, applications of the procedures used in Ref. 1 to deal with the one-input case.

II. DOUBLE VOLTERRA SERIES

Let $x_u(t)$ and $x_v(t)$ be the two inputs, and let the output $y(t)$ be given by the double Volterra series

$$y(t) = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty}{}' \frac{1}{m!n!} \int_{-\infty}^{\infty} du_1 \cdots \int_{-\infty}^{\infty} du_m \int_{-\infty}^{\infty} dv_1 \cdots \int_{-\infty}^{\infty} dv_n$$

$$\cdot g_{m;n}(u_1, \cdots, u_m; v_1, \cdots, v_n) \prod_{r=1}^{m} x_u(t - u_r) \prod_{s=1}^{n} x_v(t - v_s), \quad (1)$$

where the prime on $\sum'$ means that the term $m = n = 0$ is omitted. The product $\prod$ is understood to have the value 1 when the number of factors (m or n) is 0, and if n, say, is zero there are no v-integrations. The kernel $g_{m;n}$ is a symmetric function of $u_1, \cdots, u_m$ and of $v_1, \cdots, v_n$.

For the inputs that we shall consider, the $(m + n)$-fold Fourier transform

$$G_{m;n}(f_{u1}, \cdots, f_{um}; f_{v1}, \cdots, f_{vn}) = \int_{-\infty}^{\infty} du_1 \cdots \int_{-\infty}^{\infty} dv_n$$

$$\cdot g_{m;n}(u_1, \cdots; \cdots v_n) \exp\left[-j(u_1\omega_{u1} + \cdots + v_n\omega_{vn})\right] \quad (2)$$

plays an important role. Here $G_{0;0} \equiv 0$, $\omega = 2\pi f$, and $G_{m;n}$ is a symmetric function of $f_{u1}, \cdots, f_{um}$ and of $f_{v1}, \cdots, f_{vn}$.

Much as in Ref. 1, the "harmonic input" method can he used to determine $G_{m;n}$ from the system equations hy setting

$$\begin{aligned} x_u(t) &= \sum_{r=1}^{m} \exp(j\omega_{ur}t), \\ x_v(t) &= \sum_{s=1}^{n} \exp(j\omega_{vs}t), \end{aligned} \quad (3)$$

where the ω's are incommensurahle, and solving for the coefficient of $\exp[j(\omega_{u1} + \cdots + \omega_{um} + \omega_{v1} + \cdots + \omega_{vn})t]$ in the expansion of $y(t)$. This coefficient is equal to $G_{m;n}(f_{u1}, \cdots, f_{um}; f_{v1}, \cdots, f_{vn})$. Note that if the system output $y(t)$ is applied to the input of a linear transducer, the transducer output can also be expressed as a double Volterra series. The transducer output function corresponding to the transducer input function $G_{m;n}$ is

$$F[j(\omega_{u1} + \cdots + \omega_{vn})]G_{m;n}(f_{u1}, \cdots; \cdots f_{vn}),$$

where $F(j\omega)$ is the transfer function of the transducer.

If, say, n is zero and $m > 0$, $G_{m;0}(f_{u1}, f_{u2}, \cdots, f_{um};)$ is equal to the coefficient of $\exp[j(\omega_{u1} + \omega_{u2} + \cdots + \omega_{um})t]$ in the expansion of $y(t)$ when $x_v(t) \equiv 0$ and $x_u(t)$ is given hy (3).

There is a resemhlance between the douhle Volterra series (1) for the two-port inputs $x_u(t)$, $x_v(t)$ (Case A) and the single Volterra series

for the special input $x(t) = x_u(t) + x_v(t)$ (Case B). For Case B,

$$y(t) = \sum_{k=1}^{\infty} \frac{1}{k!} \int_{-\infty}^{\infty} du_1 \cdots \int_{-\infty}^{\infty} du_k \hat{g}_k(u_1, \cdots, u_k) \cdot \prod_{r=1}^{k} [x_u(t-u_r) + x_v(t-u_r)]. \quad (4)$$

When the product is expanded and the symmetry of $\hat{g}_k$ is used, the product can be written as

$$\sum_{m=0}^{k} \binom{k}{m} \prod_{r=1}^{m} x_u(t-u_r) \prod_{s=1}^{k-m} x_v(t-u_{m+s}).$$

Setting $n = k - m$ and $u_{m+s} = v_s$ for $s = 1, 2, \cdots, n$ carries (4) into a form which goes into (1) when $\hat{g}_{m+n}(u_1, \cdots, u_m, v_1, \cdots, v_n)$ is replaced by $g_{m;n}(u_1, \cdots, u_m; v_1, \cdots, v_n)$.

The results stated below for Case A show a similar resemblance to the corresponding Case B stated in Ref. 1. For example, when $x_u(t)$ and $x_v(t)$ are the sums of sinusoidal terms, the expression for a particular component in $y(t)$ for Case A can be obtained from the corresponding expression for Case B by replacing $\hat{G}_{m+n}$ in Case B by $G_{m;n}$ and inserting a semicolon at the appropriate place in the string of arguments [as in eq. (9) below].

III. SINUSOIDAL INPUTS

When the inputs $x_u(t)$ and $x_v(t)$ are sums of sinusoidal waves, an expression for any particular component in $y(t)$ can be obtained by extending the analysis given in Section VI-B of Ref. 1. There, eqs. [1, (138), (139), (140)] [meaning eqs. (138), (139), and (140) of Ref. 1] show that if the input $x(t)$ to a one-input system is given by

$$x(t) = \sum_{r=1}^{\mu} P_r \cos \omega_r t, \quad (5)$$

where the ω_r's are incommensurable, then the $\exp [j(N_1\omega_1 + \cdots + N_\mu\omega_\mu)t]$, $N_r \geqq 0$, component of $y(t)$ is

$$\exp [j(N_1\omega_1 + \cdots + N_\mu\omega_\mu)t] \sum_{l_1=0}^{\infty} \cdots \sum_{l_\mu=0}^{\infty} \prod_{r=1}^{\mu} \left[\frac{(P_r/2)^{N_r+2l_r}}{(N_r+l_r)!\, l_r!} \right] \cdot G_n[(f_1)_{N_1+l_1}, (-f_1)_{l_1}, \cdots, (f_\mu)_{N_\mu+l_\mu}, (-f_\mu)_{l_\mu}], \quad (6)$$

where $G_0 \equiv 0$, $(f_\sigma)_k$ denotes the string of k arguments $f_\sigma, f_\sigma, \cdots, f_\sigma,$

and the subscript n on G has the value

$$n = \sum_{r=1}^{\mu} (N_r + 2l_r). \tag{7}$$

Here the notation of Ref. 1 has been changed to bring the statement of (6) in line with the notation used in the present paper.

Metbods of computing (6) when the G_n's are constants, i.e., are independent of frequency, bave been considered by several writers (see Kroupa[5] and Sea and Vacroux[6]).

To state tbe generalization of (6) let

$$x_u(t) = \sum_{r=1}^{\mu} P_r \cos \omega_r t, \qquad x_v(t) = \sum_{r=\mu+1}^{\lambda} P_r \cos \omega_r t, \tag{8}$$

where tbe ω_r's are incommensurable, $\omega_r = 2\pi f_r$, $\lambda = \mu + \nu$, and μ and ν are positive integers. Then the $\exp [j(N_1\omega_1 + \cdots + N_\lambda\omega_\lambda)t]$, $N_r \geqq 0$, component in $y(t)$ is

$$\exp [j(N_1\omega_1 + \cdots + N_\lambda\omega_\lambda)t] \sum_{l_1=0}^{\infty} \cdots \sum_{l_\lambda=0}^{\infty} \prod_{r=1}^{\lambda} \left[\frac{(P_r/2)^{N_r+2l_r}}{(N_r + l_r)!\, l_r!} \right]$$
$$G_{m:n}[(f_1)_{N_1+l_1}, (-f_1)_{l_1}, (f_2)_{N_2+l_2}, \cdots, (f_\mu)_{N_\mu+l_\mu}, (-f_\mu)_{l_\mu};$$
$$\cdot (f_{\mu+1})_{N_{\mu+1}+l_{\mu+1}}, \cdots, (f_\lambda)_{N_\lambda+l_\lambda}, (-f_\lambda)_{l_\lambda}]. \tag{9}$$

Here $G_{0:0} \equiv 0$, and if l or $N + l$ are 0 the corresponding arguments do not appear in $G_{m:n}$. The values of m and n are

$$m = \sum_{r=1}^{\mu} (N_r + 2l_r), \qquad n = \sum_{r=\mu+1}^{\lambda} (N_r + 2l_r). \tag{10}$$

The semicolon in the subscript of $G_{m:n}$ differs in meaning from tbe semicolon used in [1, (139), (140)]. The notation $(f_\sigma)_k$ is the same as that in (6) and in [1, (169)]. The series (9) may either converge or diverge, depending on the P's and G's.

Changing tbe signs of ω_1 and f_1 in (9) carries (9) into the expression for tbe $\exp [j(-N_1\omega_1 + N_2\omega_2 + \cdots + N_\lambda\omega_\lambda)t]$ component in $y(t)$, etc. [see tbe discussion below eq. (5) in Ref. 1]. Wben some of tbe ω_r's are commensurable, some of the components in $y(t)$ coalesce and can be treated by the method used in [1, (6), (7)].

To examine the case in which $x_u(t)$ contains a dc component, let f_1 and ω_1 tend to 0 in (8) and (9). Then P_1 is the dc component of $x_u(t)$ and tbe $\exp [j(N_2\omega_2 + \cdots + N_\lambda\omega_\lambda)t]$ component of $y(t)$ is the result of the coalescence (as $f_1 \to 0$) of (i) the components $\exp [j(N_1\omega_1$

$+ N_2\omega_2 + \cdots + N_\lambda\omega_\lambda)t]$ for $N_1 = 0, 1, 2, \cdots, \infty$ and (ii) $\exp [j(-N_1\omega_1 + N_2\omega_2 + \cdots + N_\lambda\omega_\lambda)t]$ for $N_1 = 1, 2, \cdots, \infty$. When (9) and (9) with $-f_1$ in place of f_1 are summed over the values of N_1, the double sum with respect to l_1 and N_1 can be reduced to a single sum by setting $k = N_1 + 2l_1$ and using the binomial theorem. The desired component, namely $\exp [j(N_2\omega_2 + \cdots + N_\lambda\omega_\lambda)t]$, in $y(t)$ when $x_u(t) = P_1 + P_2 \cos \omega_2 t + \cdots + P_\mu \cos \omega_\mu t$ and $x_v(t)$ is given by (8) is found to be

$$\exp [j(N_2\omega_2 + \cdots + N_\lambda\omega_\lambda)t] \sum_{k=0}^{\infty} \sum_{l_2=0}^{\infty} \cdots \sum_{l_\lambda=0}^{\infty} \frac{P_1^k}{k!} \prod_{r=2}^{\lambda} \left[\frac{(P_r/2)^{N_r+2l_r}}{(N_r + l_r)!\, l_r!} \right]$$
$$\cdot G_{m;n}[(0)_k, (f_2)_{N_2+l_2}, (-f_2)_{l_2}, \cdots; \cdots, (f_\lambda)_{N_\lambda+l_\lambda}, (-f_\lambda)_{l_\lambda}]. \quad (11)$$

Here n is given by (10), and $m = k + (N_2 + 2l_2) + \cdots + (N_\mu + 2l_\mu)$ when $\mu \geqq 2$ and $m = k$ when $\mu = 1$.

Equation (9) can be generalized to the case of three or more input ports in a straigbtforward way.

IV. $x_u(t)$ GAUSSIAN AND $x_v(t) = P \cos pt$

The case $x_v(t) = P \cos pt$ and $x_u(t) = I(t)$, where $I(t)$ is a stationary, zero-mean, Gaussian noise having the two-sided power spectrum $W_I(f)$, can be handled in much the same way as was the case $x(t) = I(t) + P \cos pt$ discussed in Section VII-C of Ref. 1.

The discrete sinusoidal components in $y(t)$ are given by the ensemble average

$$\langle y(t) \rangle = \sum_{n=-\infty}^{\infty} c_n \exp (jnpt), \quad (12)$$

where

$$c_n = \sum_{\sigma=0}^{\infty} \frac{(P/2)^{2\sigma+|n|}}{(\sigma + |n|)!\, \sigma!} S_{n,\sigma,0}(; f_p),$$

$$S_{n,\sigma,k}(f_1, \cdots, f_k; f_p) = \sum_{\nu=0}^{\infty} \frac{Q_\nu[W_I(f')]}{\nu!\, 2^\nu} G_{2\nu+k;2\sigma+|n|}[f_1', -f_1', \cdots, f_\nu',$$
$$-f_\nu', f_1, f_2, \cdots, f_k; (s_n f_p)_{\sigma+|n|}, (-s_n f_p)_\sigma]. \quad (13)$$

Here $2\pi f_p = p$, $s_n = 1$ for $n \geqq 0$, $s_n = -1$ for $n < 0$, and as in (6), $(s_n f_p)_\sigma$ denotes a sequence of σ arguments, all equal to $s_n f_p$. As explained in connection with [1, (145)], $Q_\nu[W_I(f')]$ denotes a ν-fold integration with respect to $f_1', \cdots, f_\nu'$ witb limits $\pm\infty$. The integrand is $W_I(f_1') \cdots W_I(f_\nu')$ times the function [in (13) the function is G] of $f_1', \cdots, f_\nu'$ represented by all of tbe terms lying to tbe right of $Q_\nu[W_I(f')]$. $Q_0[W_I(f')]$ denotes tbe identity operator.

The two-sided power spectrum of $y(t)$ is

$$W_y(f) = \sum_{n=-\infty}^{\infty} |c_n|^2 \delta(f - nf_p) + \sum_{k=1}^{\infty} \frac{Q_k[W_I(f)]}{k!} \sum_{n=-\infty}^{\infty} \delta(f - f_1 - \cdots - f_k - nf_p) \cdot \left| \sum_{\sigma=0}^{\infty} \frac{(P/2)^{2\sigma+|n|}}{(\sigma + |n|)!\sigma!} S_{n,\sigma,k}(f_1, \cdots, f_k; f_p) \right|^2. \quad (14)$$

Replacing $G_{m:n}$ by G_{m+n} in eq. (13) for S and substituting in (14) gives the expression [1, (175)] for $W_y(f)$ in the single input port case $x(t) = I(t) + P \cos pt$. There is a corresponding similarity between the one input port formula [1, (16)] when the two-port expression (14) for $W_y(f)$ is written out.

V. EXAMPLE—COMPUTATION OF $G_{1:1}$

Consider the circuit shown in Fig. 1. The admittance $H(f)$ is linear but the resistor R and capacitor C are nonlinear. The voltage across R is

$$\alpha I_u + \beta I_u^2 \quad (15)$$

and the capacitance of C depends upon the charge Q, the capacitance being $a + bQ$. The output of interest is the voltage $y(t)$ across C:

$$\begin{aligned} Q &= (a + bQ)y, \\ I_u + I_v &= I = dQ/dt. \end{aligned} \quad (16)$$

The current $I_v(t)$ is given by

$$I_v(t) = \int_{-\infty}^{\infty} h(u)[x_v(t-u) - y(t-u)]du, \quad (17)$$

where $h(u)$ is the Fourier transform of $H(f)$.

Elimination of I_v leads to the circuit equations

$$\begin{aligned} x_u &= \alpha I_u + \beta I_u^2 + y, \\ dQ/dt &= I_u + \int_{-\infty}^{\infty} h(u)[x_v(t-u) - y(t-u)]du, \\ Q &= (a + bQ)y. \end{aligned} \quad (18)$$

The G's corresponding to y can be obtained from (18) by the harmonic input method. In using this method it is convenient to work with the notation $z_k = \exp(j\omega_k t)$ where the ω's are incommensurable.

In order to get $G_{1;0}$ $(f_1;)$ we set $x_u = z_1$, $x_v = 0$, $y = c_1 z_1 +$ higher harmonics, $I_u = i_1 z_1 + \cdots$, and $Q = q_1 z_1 + \cdots$. The harmonic input

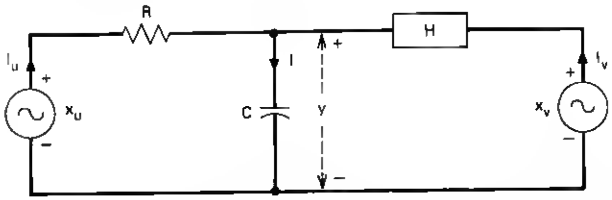

Fig. 1—Circuit with input voltages $x_u(t)$, $x_v(t)$, output voltage $y(t)$, and nonlinear R and C.

method states that $G_{1;0}\ (f_1;)$ is equal to c_1. Suhstituting in (18) and equating coefficients of z_1 gives

$$\begin{aligned} 1 &= \alpha i_1 + c_1, \\ j\omega_1 q_1 &= i_1 - H(f_1)c_1, \\ q_1 &= ac_1. \end{aligned} \tag{19}$$

Solving for c_1 gives $G_{1;0}\ (f_1;)$. Similarly, starting with $x_u = 0$ and $x_v = z_1$ gives $G_{0;1}(;f_1)$. The results are

$$\begin{aligned} G_{1;0}(f_1;) &= \left[\frac{1}{1 + \alpha H + j\omega a\alpha}\right]_{f_1}, \\ G_{0;1}(;f_1) &= \left[\frac{\alpha H}{1 + \alpha H + j\omega a\alpha}\right]_{f_1}, \end{aligned} \tag{20}$$

where the suhscript f_1 means that $f = f_1$ is to be substituted in $\omega = 2\pi f$ and in $H(f)$.

To get $G_{1;1}(f_1;f_2)$ we set $x_u = z_1$, $x_v = z_2$, and assume

$$\begin{aligned} y &= c_1 z_1 + c_2 z_2 + c_{12} z_1 z_2 + \cdots, \\ I_u &= i_1 z_1 + i_2 z_2 + i_{12} z_1 z_2 + \cdots, \\ Q &= q_1 z_1 + q_2 z_2 + q_{12} z_1 z_2 + \cdots. \end{aligned} \tag{21}$$

When (21) is substituted in the circuit equations (18), the coefficients of z_1 give the equations (19) and therefore $c_1 = G_{1;0}(f_1;)$. Similarly, $c_2 = G_{0;1}(;f_2)$. The coefficients of $z_1 z_2$ give a set of equations which, upon solving for c_{12} and using $q_1 = ac_1$, $q_2 = ac_2$, $i_2 = -c_2/\alpha$, $i_1 = \cdots$, give

$$c_{12} = 2c_1 c_2 \left[\frac{(\beta/\alpha)(H + j\omega a)_{f_1} - j(\omega_1 + \omega_2)a\alpha b}{(1 + \alpha H + j\omega a\alpha)_{f_1+f_2}}\right]. \tag{22}$$

Replacing $c_1 c_2$ by $G_{1;0}(f_1;)G_{0;1}(;f_2)$ gives the required expression for $G_{1;1}(f_1;f_2) = c_{12}$.

To get $G_{2;0}(f_1,f_2;)$ we start with $x_u = z_1 + z_2$, $x_v = 0$ and again make the suhstitutions (21) in the circuit equations (18). The coefficients of

z_1z_2 give the same set of equations as before because x_u and x_v appear only linearly in the circuit equations. We have $c_1 = G_{1;0}(f_1;)$, $i_1 = c_1[H(f_1) + j\omega_1 a]$, $q_1 = ac_1$, and $q_2 = ac_2$ as before, but now $c_2 = G_{1;0}(f_2;)$, $i_2 = c_2[H(f_2) + j\omega_2 a]$, and $c_{12} = G_{2;0}(f_1, f_2;)$.

To sum up, we have

$$G_{1;1}(f_1; f_2) = 2G_{1;0}(f_1;)G_{0;1}(;f_2) \times [\text{expression in brackets (22)}], \quad (23)$$

where $G_{1;0}$ and $G_{0;1}$ are given by (20). Expressions for $G_{2;0}(f_1, f_2;)$ and $G_{0;2}(;f_1, f_2)$ are obtained by replacing the product $G_{1;0}G_{0;1}$ in (23) by $G_{1;0}G_{1;0}$ and $G_{0;1}G_{0;1}$, respectively, and changing the bracket slightly.

To get $G_{1;2}(f_1; f_2, f_3)$ we set $x_u = z_1$, $x_v = z_2 + z_3$ and proceed along the lines used to get $G_{1;1}$, and so on.

As an example of the use of (23), suppose that $x_u = P_1 \cos \omega_1 t$ and $x_v = P_2 \cos \omega_2 t$. Then the $\exp[j(\omega_1 \pm \omega_2)t]$ component in y is, from (9),

$$\exp[j(\omega_1 \pm \omega_2)t]\left[\frac{P_1P_2}{4}G_{1;1}(f_1; \pm f_2) + \cdots\right]. \quad (24)$$

Similarly, the leading terms in the series for the components of frequency $2f_1$ and $2f_2$ are given by $G_{2;0}(f_1, f_1;)$ and $G_{0;2}(;f_2, f_2)$, respectively.

Part II. Analysis for a Simple Frequency Converter

Here the circuit shown in Fig. 2 is used as an example to show how a multi-input system can sometimes be analyzed by the single-input formulas of Ref. 1. The output of interest is the voltage $y(t)$ across the nonlinear capacitor C. Thévenin's theorem is used to replace the circuit of Fig. 2 by that of Fig. 3, and a recurrence relation is derived for the corresponding G_n's. The results are used to get an expression for the third-order distortion when Fig. 2 is regarded as the circuit for an up-converter.

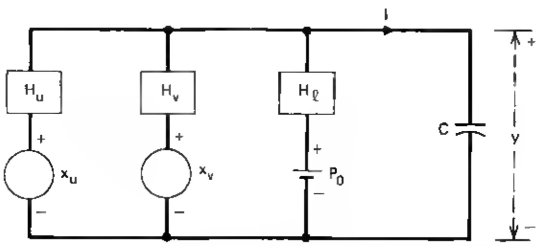

Fig. 2—Frequency converter with nonlinear capacitor C.

VI. REDUCTION OF A MULTIPLE-INPUT SYSTEM TO A SINGLE-INPUT SYSTEM

The system considered here and in the following sections is shown in Fig. 2. The admittances $H_u(f)$, $H_v(f)$, $H_l(f)$ are linear and C is the nonlinear capacitor used in Section V. The charge on C is $Q(t)$, the current $I = dQ/dt$ flows into C, the capacitance of C is $a + bQ$, and the voltage $y(t)$ across C is related to $Q(t)$ by

$$Q = (a + bQ)y. \tag{25}$$

P_0 is a biasing dc voltage.

The problem is to determine the components of $y(t)$ when

$$\begin{aligned} x_u(t) &= P_1 \cos \omega_1 t + P_2 \cos \omega_2 t, \\ x_v(t) &= P_p \cos \omega_p t, \end{aligned} \tag{26}$$

and ω_1, ω_2, and ω_p are incommensurable.

As far as $y(t)$ is concerned, the analysis of Fig. 2 can he reduced to that of the simpler circuit shown in Fig. 3. To accomplish this we apply Thévenin's theorem to the portion of Fig. 2 lying to the left of the terminals of C. As far as the exp $(j\omega_1 t)$ components of $y(t)$ and $I(t)$ are concerned, this portion of the system can be replaced by an admittance $H(f_1) = H_u(f_1) + H_v(f_1) + H_l(f_1)$ in series with the (open-circuit) voltage

$$\frac{P_1}{2} e^{j\omega_1 t} \left(\frac{H_u}{H_u + H_v + H_l} \right)_{f_1}.$$

Similar consideration of the remaining components shows that $I(t)$ and $y(t)$ can be computed from the circuit of Fig. 3 in which

$$\begin{aligned} H(f) &= H_u(f) + H_v(f) + H_l(f), \\ x(t) &= \rho_0 P_0 + \rho_1 P_1 \cos(\omega_1 t + \varphi_1) + \rho_2 P_2 \cos(\omega_2 t + \varphi_2) \\ &\qquad + \rho_p P_p \cos(\omega_p t + \varphi_p), \\ \rho_0 &= \frac{H_l(0)}{H(0)}, \qquad \rho_1 e^{j\varphi_1} = \frac{H_u(f_1)}{H(f_1)}, \\ \rho_2 e^{j\varphi_2} &= \frac{H_u(f_2)}{H(f_2)}, \qquad \rho_p e^{j\varphi_p} = \frac{H_v(f_p)}{H(f_p)}. \end{aligned} \tag{27}$$

The equation for $y(t)$, namely

$$\frac{d}{dt}\frac{ay}{1 - by} = \int_{-\infty}^{\infty} h(u)[x(t-u) - y(t-u)]du, \tag{28}$$

where $h(u)$ is the Fourier transform of $H(f)$, can he obtained hy

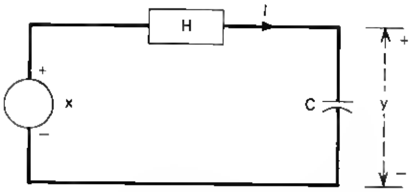

Fig. 3—Thévenin equivalent of Fig. 2.

equating two expressions for I, the one on the left being $I = dQ/dt$ in which $Q = (a + bQ)y = ay/(1 - by)$.

It is convenient to subtract out the dc components of $x(t)$ and $y(t)$ and apply the formulas of Ref. 1 to the portion $\hat{y}(t)$ of $y(t)$ which tends to 0 when $\hat{x}(t) \to 0$, $\hat{x}(t)$ being the time-varying component of $x(t)$. Therefore, in the system equation (28), we make the substitutions

$$\begin{aligned} x(t) &= x_0 + \hat{x}(t), \\ y(t) &= y_0 + \hat{y}(t), \end{aligned} \tag{29}$$

where x_0 is the dc value of $x(t)$ and y_0 is the (dc) value of $y(t)$ when $x(t) \equiv x_0$. From (27) $x_0 = \rho_0 P_0$, and substitution in (28) gives $0 = H(0)\ (x_0 - y_0)$. Assuming $H(0) \neq 0$ gives $y_0 = x_0 = \rho_0 P_0$. It should be noticed that the substitutions (29) are not strictly necessary because the dc component in $x(t)$ could be handled (at the cost of more work) by the analogue of (11).

Subtracting the result of substituting x_0 and y_0 in (28) from the result of substituting (29) in (28) and using

$$\frac{a(y_0 + \hat{y})}{1 - b(y_0 + \hat{y})} - \frac{ay_0}{1 - by_0} = \frac{\hat{a}\hat{y}}{1 - \hat{b}\hat{y}} \tag{30}$$

shows that the system to be analyzed by the single-input formulas of Ref. 1 is described by the equations

$$\frac{d}{dt}\frac{\hat{a}\hat{y}}{1 - \hat{b}\hat{y}} = \int_{-\infty}^{\infty} h(u)[\hat{x}(t - u) - \hat{y}(t - u)]du, \tag{31}$$

$$\begin{aligned} \hat{x}(t) = \rho_1 P_1 \cos(\omega_1 t + \varphi_1) &+ \rho_2 P_2 \cos(\omega_2 t + \varphi_2) \\ &+ \rho_p P_p \cos(\omega_p t + \varphi_p), \end{aligned} \tag{32}$$

where

$$\hat{a} = a/(1 - by_0)^2, \qquad \hat{b} = b/(1 - by_0). \tag{33}$$

Here $\hat{x}(t)$ and $\hat{y}(t)$ play the roles that $x(t)$ and $y(t)$ play in Ref. 1; and in the remainder of this paper the G_n's will refer to $\hat{x}(t)$ and $\hat{y}(t)$.

VII. CALCULATION OF THE G_n'S

The functions $G_1(f_1)$, $G_2(f_1,f_2),\cdots$ can be computed from (31) by the harmonic input method. A guide to the work is furnished by the resemblance of our problem to the one described by Fig. 3 of Ref. 1 and eqs. [1, (42), (43), (106)]. Expanding the left side of (31) as

$$\frac{d}{dt}\sum_{l=1}^{\infty} \hat{a}\hat{b}^{l-1}[\hat{y}(t)]^l \tag{34}$$

carries (31) into the form of [1, (106)] except for the operator d/dt. A procedure similar to the one used to deal with [1, (106)] gives

$$\begin{aligned} G_1(f_1) &= \left(\frac{H}{H+j\omega\hat{a}}\right)_{f_1}, \qquad K(f) = \frac{-2j\omega\hat{a}}{H(f)+j\omega\hat{a}}, \\ G_2(f_1, f_2) &= \hat{b}G_1(f_1)G_1(f_2)K(f_1+f_2), \\ G_3(f_1, f_2, f_3) &= \hat{b}^2G_1(f_1)G_1(f_2)G_1(f_3)K(f_1+f_2+f_3) \\ &\quad \cdot[K(f_1+f_2)+K(f_1+f_3)+K(f_2+f_3)+3], \end{aligned} \tag{35}$$

and the recurrence relation

$$G_n(f_1,\cdots,f_n) = \tfrac{1}{2}K(f_1+\cdots+f_n)\sum_{l=2}^{n}\hat{b}^{l-1}G_n^{(l)}(f_1,\cdots,f_n). \tag{36}$$

The $G_n^{(l)}$'s are the G_n's for the Volterra series for $[y(t)]^l$, and formulas for computing them are given in [1, (24) to (29)]. Equation (36) is a recurrence relation because $G_n^{(l)}$ can be expressed as the sum of products of $G_1, G_2, \cdots, G_{n-1}$. By starting with $G_1(f_1)$, the relation (36) can be used to compute $G_2, G_3, \cdots$ in succession. In the next section, (36) will be used to compute G_4.

VIII. COMPONENTS OF $y(t)$ OF FREQUENCY $f_1 + f_p$ AND $2f_1 - f_2 + f_p$

In this section we use (6) and the recurrence relation (36) to derive expressions for the $\exp[j(\omega_1+\omega_p)t]$ and $\exp[j(2\omega_1-\omega_2+\omega_p)t]$ components of $y(t)$ in Fig. 2 when (*i*) P_1, P_2, P_p are small, (*ii*) f_1 and f_2 are nearly equal, and (*iii*) $H(f)$ is zero except for frequencies lying in narrow bands about the values

$$0,\ f_1,\ f_p,\ f_u, \tag{37}$$

where f_u denotes the upper sideband frequency $f_1 + f_p$.

The component of frequency $2f_1 - f_2 + f_p$ represents a typical third-order distortion product in an up-converter when f_1 and f_2 are signal frequencies, f_p the pump frequency, and $f_u = f_1 + f_p$, $f_2 + f_p$ the output frequencies.

TABLE I—NOTATION FOR VARIOUS VALUES OF $H(f)$ AND $K(f)$

Frequency, f	$H(f)$	$K(f)$
$0,\ f_1 - f_2$	H_0	0
$f_1,\ f_2,\ 2f_1 - f_2$	H_1	K_1
$f_p,\ f_1 - f_2 + f_p$	H_p	K_p
$f_1 + f_p(=f_u),\ 2f_1 - f_2 + f_p$	H_u	K_u
outside bands	0	-2

As mentioned in Section I, the procedure we use here can be regarded as a systemization of a method used by Gardiner and Ghobrial[3] to study the distortion performance of a varactor frequency converter. In our notation, the problem they solve is that of determining the $\exp [j(2\omega_1 - \omega_2 + \omega_p)t]$ component of the charge $Q(t)$ from the system equation

$$V(t) = aQ + bQ^2 + \int_{-\infty}^{\infty} k(u)I(t-u)du. \tag{38}$$

Here $V(t)$ is the sum of three sine waves [just as $\hat{x}$ is in (32)], $I = dQ/dt$, and $k(u)$ is the Fourier transform of the linear impedance $Z(f)$ in the Thévenin equivalent of the converter circuit. It can be shown from (38) that the G_n's corresponding to $Q(t)$ can be determined by recurrence from

$$G_1(f_1) = 1/(a + j\omega Z)_{f_1},$$
$$G_n(f_1,\cdots,f_n) = \left[\frac{-b}{a + j\omega Z}\right]_{f_1+\cdots+f_n} G_n^{(2)}(f_1,\cdots,f_n). \tag{39}$$

Now we return to our own problem. From the expression (32) for the input $\hat{x}(t)$ and the leading terms in the series (6) it follows that the $\exp [j(2\omega_1 - \omega_2 + \omega_p)t]$ and $\exp [j(\omega_1 + \omega_p)t]$ components of $y(t)$ are, respectively,

$$\exp \{j[(2\omega_1 - \omega_2 + \omega_p)t + 2\varphi_1 - \varphi_2 + \varphi_p]\} \cdot (\rho_1^2\rho_2\rho_p P_1^2 P_2 P_p/32)[G_4(f_1, f_1, -f_2, f_p) + \cdots], \tag{40}$$

$$\exp \{j[(\omega_1 + \omega_p)t + \varphi_1 + \varphi_p]\} (\rho_1\rho_p P_1 P_p/4) \cdot[G_2(f_1, f_p) + (\rho_p^2 P_p^2/8)G_4(f_1, f_p, f_p, -f_p) + \cdots]. \tag{41}$$

Only one G_4 term appears in (41) because we shall assume that $\rho_1 P_1/\rho_p P_p$ and $\rho_2 P_2/\rho_p P_p$ are small compared to one.

The function G_2 is given by (35), and the remaining problem is to compute G_4 from the formula obtained by setting $n = 4$ in the recur-

rence relation (36):

$$G_4(f_1, f_2, f_3, f_4) = \tfrac{1}{2}K(f_1 + f_2 + f_3 + f_4) \cdot \sum_{l=2}^{4} \hat{b}^{l-1} G_4^{(l)}(f_1, f_2, f_3, f_4). \quad (42)$$

As explained in Ref. 1 in connection with eqs. [1, (24) to (29)], we have

$$\frac{1}{4!} G_4^{(4)}(f_1, f_2, f_3, f_4) = (1)(2)(3)(4), \quad (43)$$

$$\frac{1}{3!} G_4^{(3)}(f_1, f_2, f_3, f_4) = (1)(2)(34) + (1)(3)(24) + (1)(4)(23) + (2)(3)(14) + (2)(4)(13) + (3)(4)(12), \quad (44)$$

$$\frac{1}{2!} G_4^{(2)}(f_1, f_2, f_3, f_4) = (1)(234) + (2)(134) + (3)(124) + (4)(123) + (12)(34) + (13)(24) + (14)(23), \quad (45)$$

where we have written "(2)," for example, for $G_1(f_2)$, "(34)" for $G_2(f_3, f_4)$, "(234)" for $G_3(f_2, f_3, f_4)$, and so on.

The next step is to compute the right-hand sides of (43), (44), (45) from the expressions (35) for G_1, G_2, G_3 when only frequencies in the hands indicated hy (37) are allowed to flow. To aid in this, we introduce the notation shown in Table I for the values of the function $K(f) = -2j\omega\hat{a}/(H(f) + j\omega\hat{a})$ needed for the various G_2's and G_3's. We make the usual assumption that the admittance $H(f)$ remains constant in each hand.

We consider first the $G_4(f_1, f_1, -f_2, f_p)$ in expression (40) for the $\exp[j(2\omega_1 - \omega_2 + \omega_p)t]$ component of $y(t)$. Equation (43) gives

$$G_4^{(4)}(f_1, f_1, -f_2, f_p) = 24 G_1^2(f_1) G_1(-f_2) G_1(f_p), \quad (46)$$

where, from (35), $G_1(f) = [H/(H + j\omega\hat{a})]_f$. In eq. (44) for $G_4^{(3)}$ "(34)" now means $G_2(-f_2, f_p)$ and from the expression (35) for G_2 and Table I we get

$$(34) = \hat{b} G_1(-f_2) G_1(f_p) K(-f_2 + f_p) = \hat{b} G_1(-f_2) G_1(f_p)(-2).$$

Similarly, "(24)" means $G_2(f_1, f_p)$ and using the notation $K(f_1 + f_p) = K_u$ gives

$$(24) = \hat{b} G_1(f_1) G_1(f_p) K_u,$$

and so on. Going through all six terms for $G_4^{(3)}$ in this way carries

(44) into

$$\frac{1}{3!}G_4^{(3)}(f_1, f_1, -f_2, f_p) = \hat{b}G_1^2(f_1)G_1(-f_2)G_1(f_p) \cdot[-2 + K_u + 0 + K_u + 0 - 2].$$

Going through all seven terms in $G_4^{(2)}$ carries (45) into

$$\frac{1}{2!}G_4^{(2)}(f_1, f_1, -f_2, f_p) = \hat{b}^3G_1^2(f_1)G_1(-f_2)G_1(f_p)[K_p(K_u + 1) + K_p(K_u + 1) + (-2)(2K_u + 1) + K_1(1) + (-2)(-2) + (0)(K_u) + (K_u)(0)].$$

Substitution of the values of $G_4^{(l)}(f_1, f_2, -f_2, f_p)$, $l = 2, 3, 4$, in the expression (42) for G_4 and combining terms leads to

$$G_4(f_1, f_1, -f_2, f_p) = \hat{b}^3G_1^2(f_1)G_1(-f_2)G_1(f_p) \cdot K_u[2(K_p + 1)(K_u + 1) + K_1]. \quad (47)$$

When this is put in (40) we get the approximation we have been seeking for the third-order distortion (exp $[j(2\omega_1 - \omega_2 + \omega_p)t]$) component of $y(t)$.

The procedure used to obtain (47) can also be used to show that the G_4 in the expression (41) for the exp $[j(\omega_1 + \omega_p)t]$ component of $y(t)$ has the value

$$G_4(f_1, f_p, f_p, -f_p) = \hat{b}^3G_1(f_1)G_1^2(f_p)G_1(-f_p) \cdot K_u[2(K_1 + 1)(K_u + 1) + K_p]. \quad (48)$$

If we assume that the two series (40) and (41) converge at about the same rate, we can use (48) to get an idea of how large P_p can be before the leading term in (40) ceases to be a good approximation to the typical third-order distortion term. Thus, we expect the leading term in (40) to be a good approximation as long as the ratio

$$\tfrac{1}{8}|\rho_pP_p\hat{b}G_1(f_p)|^2[2(K_1 + 1)(K_u + 1) + K_p] \quad (49)$$

of the first two terms in (41) is small compared to unity.

Note that setting $f_2 = f_1$ and then interchanging f_1 and f_p in the expression (47) for $G_4(f_1, f_1, -f_2, f_p)$ carries it into the expression (48) for $G_4(f_1, f_p, f_p, -f_p)$. This is to be expected since $G_4(f_1, f_2, f_3, f_4)$ is a symmetric function of f_1, f_2, f_3, f_4.

REFERENCES

1. Bedrosian, E., and Rice, S. O., "The Output Properties of Volterra Systems (Nonlinear Systems with Memory) Driven by Harmonic and Gaussian Inputs," Proc. IEEE, *59*, (December 1971), pp. 1688–1707.

2. Anderson, D. R., and Leon, B. J., "Nonlinear Distortion and Truncation Errors in Frequency Converters and Parametric Amplifiers," IEEE Trans. Circuit Theory, *CT-12*, (September 1965), pp. 314–321.
3. Gardiner, J. G., and Ghobrial, S. I., "Distortion Performance of the Abrupt-Junction Current-Pumped Varactor Frequency Converter," IEEE Trans. Microwave Theory Tech., *MTT-19*, (September 1971), pp. 741–749.
4. Swerdlow, R. B., unpublished work.
5. Kroupa, V., "Amplitude of the General Intermodulation Product," Proc. IEEE, *58*, (May 1970), pp. 851–852.
6. Sea, R. G., and Vacroux, A. G., "On the Computation of Intermodulation Products for a Power Series Nonlinearity," Proc. IEEE, *57*, (March 1969), pp. 337–338.